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On Multi-fuzzy rough sets

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Abstract-Rough sets have been invented to cope with uncertainty and to deal with intelligent systems characterized by incomplete information. This paper introduces the concept of relation based multi-fuzzy rough approximation operators. Multi-fuzzy sets are approximated by a multi-fuzzy relation and hence the concept of multi-fuzzy rough sets is defined. The basic set theoretic operations such as union, intersection etc. are defined in the context of multi-fuzzy rough sets. The basic rough set properties are then explored.

Keywords: Approximation operators, Multi-fuzzy sets, Multi-fuzzy relation, Multi-fuzzy rough sets, Fuzzy-rough sets, Fuzzy sets, Rough sets.

1 Introduction

The theory of rough sets was proposed by Zdsilaw Pawlak in 1982 [4]. The theory is in a state of constant development from the date. Both paradigms address the phenomenon of non-crisp sets, notions which according to Frege, are characterized by the presence of a non-empty boundary which does encompass subjects neither belonging with certainity to the given concept nor belonging with certainity to its complement[5].

Basically rough sets embody the idea of indiscernibility between objects in a set, while fuzzy sets model the ill definition of the boundary of a sub-class of this sets. Rough sets are a calculus of partitions, while fuzzy sets are continuous generalization of set-characteristic functions. Although rough sets and fuzzy sets address distinct aspects of reasoning under uncertainity, yet both these ideas can be combined into a hybrid approach.

Dubois and Prade studied first the fuzzification problem of rough sets and proposed the concepts of rough-fuzzy sets and fuzzy-rough sets[2]. Combining both notions lead to consider rough approximations of fuzzy sets and approximation of sets by means of similarity relations or fuzzy partitions.

Multi-fuzzy sets are introduced [6, 7] as a generalization of fuzzy sets using ordinary fuzzy sets as building blocks. The notion of multi-fuzzy set provides a new method to represent some problems, which are difficult to explain in other extensions of fuzzy-set theory.

The present paper studies the concept of hybrid structures involving multi-fuzzy sets and rough sets. In the next section, we review some basic notions related to multi-fuzzy sets and rough sets. In section 3 and 4, we define rough multi-fuzzy sets and multi-fuzzy rough sets respectively. In the respective sections, the properties of each of them are also studied.

2 Preliminaries

In this section we present some basic notions related to multi-fuzzy sets and rough sets. Let U be a non-empty set called the universe of discourse. The class of all subsets of U will be denoted by P(U).

Definition 2.1. [3] Let U be a non-empty ordinary set, L a complete lattice. An L-fuzzyset on U is a mapping $A: X \to L,$ that is the family of all the L-fuzzy sets on U is just L^U consisting of all the mappings from U to L.

Definition 2.2. [6, 7] Let U be a non-empty set, N the set of all natural numbers and $\{L_i ; i \in N\}$ a family of complete lattices. A multi-fuzzy set A in U is a set of ordered sequences

$$A = \{< x, \mu_A^1(x), \mu_A^2(x), \ldots, u_A^i(x), \ldots >; x \in U\}$$

Where $u_i \in L_i^U \left(i.e., u_i : U \to L_i\right)$ for $i \in N$

If the sequences of the membership functions have only a finite number of k terms, k is called the dimension of A. Let $L_i = [0,1] (\text{for } i = 1, 2, \ldots, k),$ then the set of all

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multi-fuzzy sets in u of dimension k, is denoted by $M^k FS(U)$.

Every fuzzy set A can be represented as a multi-fuzzy set $A = <\mu_1, \mu_2>$ of dimension two. Let μ_1, μ_2 be linearly dependent with the relation $\mu_1(x) + \mu_2(x) = 1$ for every x in U, then the multi-fuzzy set represents an ordinary fuzzy set with the membership value $\mu_1(x)$. If $\mu_1(x) + \mu_2(x) \le 1$, for every $x \in U$, then the multi-fuzzy set represents an Atanassov intuitionistic fuzzy set [1]. Some basic relations and operations on $M^k FS(U)$ are defined as follows: [7] For every $A, B \in M^k FS(U)$,

a) $A \in B$ if and only if $\mu_A^i(x) \le \mu_B^i(x)$ for all $x \in U$ and for all $i = 1,2,3,\ldots,k$

b) $A = B$ if and only if $\mu_A^i(x) = \mu_B^i(x)$ for all $x \in U$ and for all $i = 1,2,3,\ldots,k$

c) $A \cup B = \{< x, \mu_A^1(x) \vee \mu_B^1(x), \mu_A^2(x) \vee \mu_B^2(x), \ldots, \mu_A^k(x) \vee \mu_B^k(x) > / x \in U\}$

d) $A \cap B = \{< x, \mu_A^1(x) \wedge \mu_B^1(x), \mu_B^1(x), \mu_A^2(x) \wedge \mu_B^2(x), \ldots, \mu_A^k(x) \wedge \mu_B^k(x) > x \in U\}$

The multi-fuzzy universe set is $1_U = \{< x,1,1,\ldots 1,\ldots > / x \in U\}$ and the multi-fuzzy empty set is $\phi_U = \{< x,0,0,\ldots > / x \in U\}$.

For any $A \in M_k FS(U)$, the complement of A denoted by ~ A is defined as, for

$$A = \{< x, \mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^k(x) > / x \in U\},$$

$$\sim A = \{< x, 1 - \mu_A^1(x), 1 - \mu_A^2(x), \ldots, 1 - \mu_A^k(x) > / x \in U\}$$

.

According to the above definitions, the following basic properties of MF sets can be easily derived:

1. $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$
2. $A \cap B \subseteq A$ and $A \cap B \subseteq B$
3. $A \subseteq B$ and $C \subseteq D \Rightarrow A \cap C \subseteq B \cap D$
4. $\sim(\sim A) = A$

Definition 2.3. Let A, B be any sets. Then a multi-fuzzy relation R from A to B is a multi-fuzzy subset of A × B.

$$R = \{< (a,b), \mu_R^1(a,b), \mu_R^2(a,b), \ldots > / a \in A, b \in B\}$$

where $\mu_R^i : A \times B \to L_i$

Now the multi-fuzzy(MF) relation R on U is a multi-fuzzy subset of U × U

$$R = \{< (x,y), \mu_R^1(x,y), \mu_R^2(x,y), \ldots > / x, y \in U\}$$

Where $\mu_R^i : U \times U \to L_i$

If $[0,1]$, the set of all MF relations on U is denoted by MFR(U × U).

Definition 2.4. [8] Let U be a non-empty and finite universe of discourse and $R \subseteq U \times U$, an arbitrary crisp relation on U. We define a set-valued function $R_s : U \to P(U)$ by

$$R_s(x) = \{y \in U / (x,y) \in R\}, \quad x \in U.$$

$R_s(x)$ is referred to as the successor neighbourhood of x with respect to (w.r.t.) R. The pair (U, R) is called a crisp approximation space. For any $A \subseteq U$, the upper and lower approximations of A w.r.t. (U,R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, are, respectively, defined as follows:

$$\overline{R} = \{x \in U | R_s(x) \subseteq A\}$$

$$\underline{R} = \{x \in U | R_s(x) \cap A \ne \phi\}$$

The pair $(\underline{R}(A), \overline{R}(A))$ is referred to as a crisp rough set and $\underline{R}, \overline{R} : P(U) \to P(U)$ are, respectively, referred to as lower and upper crisp approximation operators induced from (U, R).

The crisp approximation operators satisfy the following properties [8]:

For all $A, B \in P(U)$,

$(L1)\underline{R}(A) = \sim \overline{R}(\sim A) = \sim \overline{R}(\sim A)$

$(L2)\underline{R}(U) = U$

$(L3)\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$

$(L4)A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$

$(L5)\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$

$(U1)\overline{R}(A) = \sim \underline{R}(\sim A)$

$(U2)\overline{R}(\phi) = \phi$

$(U3)\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$

$(U4)A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$

$(U5)\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$

Properties (L1) and (U1) show that the approximation operators $\underline{R}$ and $\overline{R}$ are dual to each other. Properties with the same number may be considered as dual properties. If R is an equivalence relation on U, then the pair (U, R) is called a Pawlak approximation space and $\underline{R}(A), \overline{R}(A)$ is a Pawlak Rough set.

3. Construction of multi-fuzzy rough sets

In this section, we will introduce Multi-fuzzy (MF) rough approximation operators induced from a MF approximation space and discuss their properties.

Definition 3.1. Let U be a non-empty and finite universe of discourse and $R \in MFR(U \times U)$, the pair (U, R)is called a multi-fuzzy approximation space. For $A \in MFS(U)$, the family of all multi-fuzzy sets on U, the lower and upper approximations of A w.r.t (U, R) denoted by $\underline{R}(A)$ and $\overline{R}(A)$ are two multi-fuzzy sets and are, respectively, defined as follows:

$$\underline{R}(A) = \left\{ \begin{array}{l} < x, \mu^1_{\underline{R}(A)}(x), \mu^2_{\underline{R}(A)}(x), \ldots, \\ \mu^k_{\underline{R}(A)}(x), \ldots, \mu^k_{\underline{R}(A)}(x), \ldots / x \in U \end{array} \right\},$$

$$\overline{R}(A) = \left\{ \begin{array}{l} < x, \mu^1_{\overline{R}(A)}(x), \mu^2_{\overline{R}(A)}(x), \ldots, \\ \mu^k_{\overline{R}(A)}(x), \ldots \mu^k_{\overline{R}(A)}(x), \ldots > / x \in U \end{array} \right\},$$

Where

$$\mu^i_{\underline{R}(A)}(x) = \wedge\left\{ \vee \left(1 - \mu^i_R(x, y), \mu^i_A(y)\right) / y \in U \right\}$$

$$\mu^i_{\overline{R}(A)}(x) = \vee\left\{ \wedge \left(\mu^i_R(x, y), \mu^i_A(y)\right) / y \in U \right\}$$

$\forall i \in N$ and $\forall x \in U$

$\underline{R}(A)$ and $\overline{R}(A)$ are, respectively, called the lower and upper approximations of A w.r.t (U, R). The pair $\left(\underline{R}(A), \overline{R}(A)\right)$ is called the MF rough set of A w.r.t. (U, R) and $\underline{R}, \overline{R} : MFS(U) \rightarrow MFS(U)$ are referred to as lower and upper multi-fuzzy rough approximation operators, respectively.

Theorem 3.2. Let U be a non-empty and finite universe of discourse and $R, R_1, R_2 \in MFR(U \times U)$. Then the lower and upper approximation operators in definition 4.1 satisfy the following properties:

$\forall A, B \in MFS(U)$,

$(MFL1)\underline{R}(A) = \sim \overline{R}(\sim A)$

$(MFL2)\underline{R}(U) = 1_U$

$(MFL3)\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$

$(MFL4)A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$

$(MFL5)\underline{R}(A \cup B) \supseteq \underline{R}(A) \cap \underline{R}(B)$

$(MFL6)R_1 \subseteq R_2 \Rightarrow \underline{R}_1(A) \subseteq \underline{R}_2(A)$

$(MFU1)\overline{R}(A) = \sim \underline{R}(\sim A)$

$(MFU2)\overline{R}(\phi) = \phi$

$(MFU3)\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$

$(MFU4)A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$

$(MFU5)\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$

$(MFU6)R_1 \subseteq R_2 \Rightarrow \overline{R}_1(A) \subseteq \overline{R}_2(A)$

Proof:

$(MFL1) Let A = \left\{ < x, \mu^1_A(x), \mu^2_A(x), \ldots, \mu^k_A(x) > x \in U \right\}$

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Then

$$\sim A = \{< x, 1-\mu_A^1(x), 1-\mu_A^2(x), \dots, 1-\mu_A^k(x) > / x \in U\}$$

and

$$\overline{R}(\sim A) = \{< x, \mu^1_{\overline{R}(\sim A)}(x), \mu^2_{\overline{R}(\sim A)}(x),\dots,\mu^k_{\overline{R}(\sim A)}(x) > / x\in U\}$$

where

$$\mu^i_{\overline{R}(\sim A)}(x) = Max\{Min(\mu_R^i(x,y), 1-\mu_A^i(y)) / y\in U\} \forall x\in U$$

and $\forall i = 1,2,\dots,k$.. Now

$$\sim \overline{R}(\sim A) = \{< x, 1-\mu^1_{\overline{R}(\sim A)}(x), 1-\mu^2_{\overline{R}(\sim A)}(x),\dots, 1-\mu^k_{\overline{R}(\sim A)}(x) > / x\in U\}$$

where

$$1-\mu^i_{\overline{R}(\sim A)}(x) = 1 - Max\{Min(\mu_R^i(x,y),1-\mu_A^i(y))/y\in U\}$$

$$= 1-[1-Min\{1-Min\{(u_R^i(x,y),1-\mu_A^i(y))/y\in U\}]$$

$$= Min\{Max(1-\mu_R^i(x,y),\mu_A^i(y))/y\in U\}$$

$=\mu^i_{\underline{R}(A)}(x)\forall x\in U$ and for $i=1,2,\dots,k$

Thus $\underline{R}(A) = \sim\overline{R}(\sim A)$.

Similarly (MFU1) can be proved. This shows that the MF rough approximation operators $\underline{R}$ and $\overline{R}$ are dual to each other.

$$A\cup B = \{< x, \mu_A^1(x)\vee\mu_B^1(x), \mu_A^2\vee\mu_B^2(x),\dots,\mu_A^k(x)\vee\mu_B^k(x) > / x\in U\}$$

.

Then

$$\overline{R}(A\cup B) = \left\{ \begin{array}{l} < x, \mu^1_{\overline{R}(A\cup B)}(x), \mu^2_{\overline{R}(A\cup B)}(x),\dots, \\ \mu^k_{\overline{R}(A\cup B)}(x) > / \in U \end{array}\right\}$$

where

$$\mu^i_{\overline{R}(A\cup B)}(x) = Max\{Min(\mu_R^i(x,y),\mu_A^i(y)\vee\mu_B^i(y))/y\in U\},$$

$x\in U$ and $i=1,2,\dots,k.$

$$\overline{R}(A)\cup\overline{R}(B) = \left\{ \begin{array}{l} < x, \mu^1_{\overline{R}(A)}(x)\vee\mu^1_{\overline{R}(B)}(x), \mu^2_{\overline{R}(A)}(x)\vee\mu^2_{\overline{R}(B)}(x),\dots \\ , \mu^k_{R(A)}(x)\vee\mu^k_{\overline{R}(B)}(x) > / x\in U \end{array}\right\}$$

where

$$\mu^k_{R(A)}(x)\vee\mu^k_{R(B)}(x) = Max\left\{Max\left\{\begin{array}{l} Min(\mu_R^i(x,y),\mu_A^i(y)), \\ Min(\mu_R^i(x,y),\mu_B^i(y))/y\in U \end{array}\right\}\right\}$$

Claim:- For $i=1,2,\dots,k$

$$Min(\mu_R^i(x,y), Max(\mu_A^i(y),\mu_B^i(y))) =$$
$$Max[Min(\mu_R^i(x,y),\mu_A^i(y)), Min(\mu_R^i(x,y),\mu_B^i(y))]$$

Case 1:- $Min(\mu_R^i(x,y),\mu_A^i(y)) = \mu_R^i(x,y)$

$$Min(\mu_R^i(x,y),\mu_B^i(y))=\mu_R^i(x,y)$$
$$Min(\mu_R^i(x,y), Max(\mu_A^i(y),\mu_B^i(y))) = \mu_R^i(x,y)$$
$$Max[Min(\mu_R^i(x,y),\mu_A^i(y)), Min(\mu_R^i(x,y),\mu_B^i(y))]$$
$$= Max(\mu_R^i(x,y),\mu_R^i(x,y)) = \mu_R^i(x,y)$$

Case 2:- $Min(\mu_R^i(x,y),\mu_A^i(y))=\mu_R^i(x,y)$

$$Min(\mu_R^i(x,y),\mu_B^i(y))=\mu_B^i(y)$$
$$Min(\mu_R^i(x,y),Max(\mu_A^i(y)\mu_B^i(y)))$$
$$=Min(\mu_R^i(x,y),\mu_A^i(y))=\mu_R^i(x,y)$$
$$Max[Min(\mu_R^i(x,y),\mu_A^i(y)), Min(\mu_R^i(x,y),\mu_B^i(y))]$$
$$=Max=(\mu_R^i(x,y),\mu_B^i(y))=\mu_R^i(x,y)$$

Case 3:- $Min(\mu_R^i(x,y),\mu_A^i(y))=\mu_A^i(y)$

$$Min(\mu_R^i(x,y),\mu_B^i(y))=\mu_R^i(x,y)$$
$$Min(\mu_R^i(x,y),Max(\mu_A^i(y),\mu_B^i(y)))$$
$$=Min(\mu_R^i(x,y),\mu_B^i(y))=\mu_R^i(x,y)$$
$$Max[Min(\mu_R^i(x,y),\mu_A^i(y)), Min(\mu_R^i(x,y),\mu_B^i(y))]$$
$$=Max(\mu_A^i(y),\mu_R^i(x,y))=\mu_R^i(x,y)$$

Case 4:- $Min(\mu_R^i(x,y),\mu_A^i(y))=\mu_A^i(y)$

$$Min(\mu_R^i(x,y),\mu_B^i(y))=\mu_B^i(y)$$
$$Min(\mu_R^i(x,y),Max(\mu_A^i(y),\mu_B^i(y)))=Max(\mu_A^i(y),\mu_B^i(y))$$
$$Max[Min(\mu_R^i(x,y),\mu_A^i(y)), Min(\mu_R^i(x,y),\mu_B^i(y))]$$
$$=Max(\mu_A^i(y),\mu_B^i(y))$$

Thus in all cases,

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$$Min\left(\mu_R^i(x,y), Max\left(\mu_A^i(y), \mu_B^i(y)\right)\right)$$
$$= Max\left[Min\left(\mu_R^i(x,y), \mu_A^i(y)\right), Min\left(\mu_R^i(x,y), \mu_B^i(y)\right)\right]$$

Thus our claim holds. Taking maximum over all $y \in U$, we get

$$\mu^i_{\overline{R}(A\cup B)}(x) = \mu^i_{\overline{R}(A)}(x) \vee \mu^i_{\overline{R}(B)}(x)$$

$\forall x \in U$ and $i = 1,2,\ldots,k$

Thus we get $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$

Similarly (MFL3) can be proved

The other properties also can be proved in a similar manner.

Example 3.3. Let $U = \{x_1, x_2, x_3\}$ and

$$R = \{<(x_1,x_1),1,1,1,1>, <(x_2,x_2),1,1,1,1>$$
$$,<(x_3,x_3),1,1,1,1>, <(x_1,x_2),.1,.2,.3,.4>$$
$$,<(x_1,x_3),0,.3,.3,.1>, <(x_2,x_1),.1,.2,.3,.4>$$
$$,<(x_2,x_3),0,.2,.4,.1>, <(x_3,x_1),0,.3,.3,.1>, <(x_3,x_2),0,.2,.4,.1>\}$$

If

$$A = \{<x_1,.4,.5,.3,.6>, <x_2,.2,.3,.4,.3>, <x_3,0,.1,.7,.8>\},$$

then by definition.

$$\underline{R}(A) = \{<x_1,.4,.5,.3,.6>, <x_2,.2,.3,.4,.3>, <x_3,0,.1,.6,.8>\}$$

$$\overline{R}(A) = \{<x_1,.4,.5,.3,.6>, <x_2,.2,.3,.4,.4>, <x_3,.4,.3,.7,.8>\}$$

4 Conclusion

This paper deals with relation based multi-fuzzy rough approximation operators. Besides giving the basic definitions, the fundamental properties of approximation operators are also proved. The introduced structure has the nice properties of both constituents and hence more relevant.

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